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Biorthogonal polynomials and total positive functions

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Abstract

Recently Ercolani and McLaughlin proved that the zeros of the biorthogonal polynomials with the weight function $w(x, y) = \exp(-V(x) - W(y) - 2\tau xy)$ are all real and distinct, and Mehta has extended their argument to the weight function $w(x, y) = \mathrm{e}^{-x-y}/(x + y)$ and to the more general case of the convolution $(w_1 * w_2 * \cdots * w_m)(x, y)$, where w_i are functions of the same form as above. Using the concept of total positive and sign-regular functions, we further extend the argument to a large class of weight functions. Many examples are presented, including several whose pair of biorthogonal polynomials turn out to come from different families of classical orthogonal polynomials.

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1. Introduction

The biorthogonal polynomials considered in this paper are two families of polynomials $\{p_n\}$ and $\{q_n\}$ related to the weight function of two variables, $w(x, y)$, by the following biorthogonal relation:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} p_n(x) q_m(y) w(x, y) \,\mathrm{d}x\, \mathrm{d}y = h_n \delta_{m,n} \qquad h_n \neq 0 \tag{1.1}$$

where p_n and q_n are polynomials of degree exactly n. These polynomials are studied in association with the random matrix theory.

Let $w(x, y)$ be a weight function defined on $X \times Y$, where X and Y are Borel sets of $\mathbb{R}$, such that all its moments

$$m_{i,j} = \int_X \int_Y x^i y^j w(x, y) \,\mathrm{d}x\, \mathrm{d}y \tag{1.2}$$

and the determinant of the moments

$$D_n = \det(m_{i,j})_{i,j=0}^n \neq 0 \qquad n \geqslant 0 \tag{1.3}$$

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where dx and dy are the Lebesgue measure on X and Y; if either X or Y is discrete, we take the measure as the counting measure. Then the biorthogonal polynomials $\{p_n\}$ and $\{q_n\}$ exist, and they are unique if we assume that these polynomials are monic. A polynomial is monic if its coefficient of the highest degree term is 1. Just like the usual orthogonal polynomials, they can be expressed as determinants:

$$p_n(x) = \det \begin{bmatrix} m_{0,0} & m_{0,1} & \dots & m_{0,n-1} & 1 \\ m_{1,0} & m_{1,1} & \dots & m_{1,n-1} & x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n,0} & m_{n,1} & \dots & m_{n,n} & x^n \end{bmatrix}$$

and

$$q_n(x) = \det \begin{bmatrix} m_{0,0} & m_{0,1} & \dots & m_{0,n} \\ m_{1,0} & m_{1,1} & \dots & m_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,0} & m_{n-1,1} & \dots & m_{n-1,n} \\ 1 & x & \dots & x^n \end{bmatrix}.$$

Recently, Ercolani and McLaughlin [2] showed that these polynomials exist for the weight functions

$$w(x, y) = \exp(-V(x) - W(y) - 2\tau xy) \qquad x, y \in \mathbb{R} \tag{1.4}$$

in which V and W are smooth functions with polynomial growth at ∞ and τ is a nonzero constant such that all the moments of w exist for τ in some fixed neighbourhood of zero, and they studied various properties of these polynomials. Among many other things, they proved that the zeros of biorthogonal polynomials p_n and q_n with respect to the weight function (1.4) are all real and distinct.

In a follow-up paper, Mehta [4] showed that the argument of Ercolani and McLaughlin can be applied to other weight functions and proved that the zeros of biorthogonal polynomials are real and distinct for the weight function

$$(w_1 * w_2 * \cdots * w_m)(x, y) = \int_{\mathbb{R}^{m-1}} w_1(x, z_1) w_2(z_1, z_2) \cdots w_m(z_{m-1}, y)\, \mathrm{d}z_1 \cdots \mathrm{d}z_{m-1} \tag{1.5}$$

where w_k are the weight functions

$$w_k(x, y) = \exp(-V_k(x) - W_k(y) - 2\tau_k xy) \qquad x, y \in \mathbb{R}$$

in which V_k, W_k and τ_k are as in (1.4) such that all the moments of the function $(w_1 * w_2 * \cdots * w_m)(x, y)$ exist. Furthermore, Mehta proved that the same holds for the weight function

$$w(x, y) = \mathrm{e}^{-x-y}/(x + y) \qquad 0 \leqslant x, y < \infty \tag{1.6}$$

and its convolution extensions.

The purpose of this paper is to show that using the concept of total positive functions and sign-regular functions, the argument in [2] can be extended to study biorthogonal polynomials for rather general weight functions and to present a large number of examples. The main results are stated and proved in the next section. In section 3 we give various examples of the weight functions for which biorthogonality exists. In particular, we show that for some weight functions the biorthogonal polynomials can come from families of classical orthogonal polynomials, including both continuous and discrete families, although p_n and q_n are often from different families of orthogonal polynomials.

2. Main results

The total positive and sign-regular functions are studied in detail in [3]. We recall the basic definitions. Let X and Y be the two sets of $\mathbb{R}$. A real function $w(x, y)$ of two variables defined on $X \times Y$ is said to be *total positive of order r* (abbreviated TP$_r$) if for all

$$x_1 < x_2 < \cdots < x_n \qquad y_1 < y_2 < \cdots < y_n \qquad x_i \in X \quad y_j \in Y$$

and for all positive integers $n \leqslant r$, we have the inequality

$$\det[w(x_j, y_k)]_{j,k=0,\ldots,n} := \det \begin{bmatrix} w(x_1, y_1) & w(x_1, y_2) & \ldots & w(x_1, y_n) \\ w(x_2, y_1) & w(x_2, y_2) & \ldots & w(x_1, y_n) \\ \vdots & \vdots & & \vdots \\ w(x_n, y_1) & w(x_n, y_2) & \ldots & w(x_n, y_n) \end{bmatrix} \geqslant 0.$$

If strict inequality holds, then we say that w is *strictly total positive of order r* (STP$_r$). More generally, a function $w(x, y)$ is called *sign-regular of order r* (SR$_r$), if there exists a sequence of numbers ε_n, each either 1 or -1 such that

$$\varepsilon_n \det[w(x_j, y_k)]_{j,k=0,\ldots,n} \geqslant 0 \qquad 1 \leqslant n \leqslant r$$

and it is called *strict sign-regular* (SSR$_r$) if strict inequalities hold. If $r = \infty$, then we simply write TP, STP, SR and SSR. Note that for $r = \infty$, X and Y must be infinite sets of $\mathbb{R}$.

Intimately connected with the concept of total positivity is the Chebyshev system of functions. A sequence of continuous functions $\phi_0(x), \ldots, \phi_n(x)$ is a Chebyshev system on $a < x < b$ if, for any set of real numbers $c_0, \ldots, c_n$, not all zero, the function $\sum_{k=0}^{n} c_k \phi_k(x)$ does not vanish more than n times on the interval (a, b). An equivalent definition is that for all $a < x_0 < x_1 < \cdots < x_n < b$ the determinant

$$\det[\phi_j(x_k)]_{j,k=0,\ldots,n} := \det \begin{bmatrix} \phi_0(x_0) & \phi_0(x_1) & \ldots & \phi_0(x_n) \\ \phi_1(x_0) & \phi_1(x_1) & \ldots & \phi_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_n(x_0) & \phi_n(x_1) & \ldots & \phi_n(x_n) \end{bmatrix}$$

never vanishes, and therefore maintains a fixed sign.

Theorem 2.1. *Let $w(x, y)$ be* STP *or* SRP *such that all moments $m_{i,j}$ of w exist. Then the monic biorthogonal polynomials p_n and q_n are uniquely determined by the relation*

$$\int_X \int_Y p_n(x) q_m(y) w(x, y) \, \mathrm{d}x \, \mathrm{d}y = h_n \delta_{m,n} \qquad h_n \neq 0. \tag{2.1}$$

Moreover, all the zeros of p_n and q_n are real, distinct and lie in X and Y, respectively.

Proof. In order to show the existence of p_n and q_n, we need to show that the moment matrix D_n is nonzero for all $n \geqslant 1$. Using the multi-linearity of the determinant, we have

$$\begin{aligned} D_n &= \det\left[\int_X \int_Y x^j y^k w(x, y) \, \mathrm{d}x \, \mathrm{d}y\right]_{j,k=0,\ldots,n} \\ &= \int_{X^{n+1}} \int_{Y^{n+1}} \det\left[x_j^j y_k^k w(x_j, y_k)\right]_{j,k=0,\ldots,n} \mathrm{d}x_0 \cdots \mathrm{d}x_n \, \mathrm{d}y_0 \cdots \mathrm{d}y_n \\ &= \int_{X^{n+1}} \int_{Y^{n+1}} \prod_{l=0}^{n} x_l^l \prod_{m=0}^{n} y_m^m \det[w(x_j, y_k)]_{j,k=0,\ldots,n} \, \mathrm{d}x_0 \cdots \mathrm{d}x_n \, \mathrm{d}y_0 \cdots \mathrm{d}y_n \end{aligned}$$

$$= \sum_{\sigma} \int_{x_{\sigma(0)} < x_{\sigma(1)} < \cdots < \cdots x_{\sigma(n)}} \sum_{\tau} \int_{y_{\tau(0)} < y_{\tau(1)} < \cdots < y_{\tau(n)}} \prod_{l=0}^{n} x_l^l \prod_{m=0}^{n} y_m^m$$
$$\times \det [w(x_j, y_k)]_{j,k=0,\ldots,n} \, \mathrm{d}x_0 \cdots \mathrm{d}x_n \, \mathrm{d}y_0 \cdots \mathrm{d}y_n$$

where the summations are taken over all permutations, σ and τ, of the integers $\{0, 1, \ldots, n\}$. Changing the summation index and using the fact that

$$\det \left[w\left(x_{\sigma(j)}, y_{\tau(k)}\right)\right]_{j,k=0,\ldots,n} = (-1)^{\sigma}(-1)^{\tau} \det [w(x_j, y_k)]_{j,k=0,\ldots,n}$$

we then have

$$D_n = \int_{x_0 < x_1 < \cdots < x_n} \int_{y_0 < y_1 < \cdots < x_n} \sum_{\sigma} (-1)^{\sigma} \prod_{l=0}^{n} \left(x_{\sigma^{-1}(l)}\right)^l \sum_{\tau} (-1)^{\tau} \prod_{m=0}^{n} \left(y_{\sigma^{-1}(m)}\right)^m$$
$$\times \det[w(x_j, y_k)]_{j,k=0,\ldots,n} \, \mathrm{d}x_0 \cdots \mathrm{d}x_n \, \mathrm{d}y_0 \cdots \mathrm{d}y_n$$
$$= \int_{x_0 < x_1 < \cdots < x_n} \int_{y_0 < y_1 < \cdots < x_n} \prod_{j<k} (x_k - x_j) \prod_{j<k} (y_k - y_j)$$
$$\times \det[w(x_j, y_k)]_{j,k=0,\ldots,n} \, \mathrm{d}x_0 \cdots \mathrm{d}x_n \, \mathrm{d}y_0 \cdots \mathrm{d}y_n$$

where the last equals sign follows from the fact that

$$\sum_{\sigma} (-1)^{\sigma} \prod_{l=0}^{n} \left(x_{\sigma^{-1}(l)}\right)^l = \det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_0 & x_1 & \ldots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & \ldots & x_n^n. \end{bmatrix} = \prod_{j<k} (x_k - x_j).$$

Consequently, the fact that w is STP or SSR shows that $D_n \neq 0$. □

Theorem 2.2. *Let X and Y be two open intervals. Let $w(x, y)$ be a function defined on $X \times Y$ such that w is* STP *or* SRP *and all moments $m_{i,j}$ of w exist. Then all zeros of p_n and q_n are real, distinct and lie inside X and Y, respectively.*

Proof. For each non-negative integer m, let ϕ_m and ψ_m be functions defined by

$$\phi_m(x) = \int_Y y^m w(x, y) \, \mathrm{d}y \qquad x \in X$$
$$\psi_m(y) = \int_X x^m w(x, y) \, \mathrm{d}x \qquad y \in Y.$$

We prove that the family of functions $\{\phi_0, \phi_1, \ldots, \phi_m\}$ forms a Chebyshev system on X. For this we need to show that for all $x_0 < x_1 < \cdots < x_m, x_i \in X$, the determinant $\det [\phi_j(x_k)]_{j,k=0,\ldots,m}$ never vanishes. However, a calculation similar to the proof of the previous theorem shows that

$$\det[\phi_j(x_k)]_{j,k=0,\ldots,m} = \det \left[\int_Y y^j w(x_k, y) \, \mathrm{d}y \right]_{j,k=0,\ldots,m}$$
$$= \int_{Y^{m+1}} \det \left[y_j^j w(x_k, y_j)\right]_{j,k=0,\ldots,m} \mathrm{d}y_0 \cdots \mathrm{d}y_m$$
$$= \int_{y_0 < y_1 < \cdots < y_m} \det[w(x_k, y_j)]_{j,k=0,\ldots,m} \prod_{i<l} (y_l - y_i) \, \mathrm{d}y_0 \cdots \mathrm{d}y_m.$$

Again, the fact that w is STP or SSR shows that the $\det [\phi_j(x_k)]_{j,k=0,\ldots,m}$ is nonzero. Similarly, the family $\{\psi_0, \psi_1, \ldots, \psi_m\}$ forms a Chebyshev system on Y.

We now prove the property on the zeros of p_n; the proof for q_n is similar. The biorthogonal relation (2.1) implies that

$$\int_X p_n(x)\phi_m(x)\,\mathrm{d}x = \int_X\int_Y p_n(x)y^m w(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0 \qquad 0 \leqslant m \leqslant n-1.$$

Suppose $z_1 < \cdots < z_m$, $m < n$, are all distinct zeros of p_n having odd multiplicity in X. By the definition of the Chebyshev system, there exists a function $\phi(x) = \sum_{j=0}^m c_k\phi_k(x)$ that vanishes at $z_1, \ldots, z_m$ and ϕ has no other zeros in X. This, however, implies that

$$\int_X p_n(x)\sum_{j=0}^m c_k\phi_k(x)\,\mathrm{d}x \neq 0$$

which contradicts the orthogonal relation, so that p_n has exactly n distinct zeros in X. □

In the proof of this theorem we need X and Y to be open intervals to use the property of the Chebyshev system. We note that to show the zeros of p_n are distinct and in X, we only need X to be an interval. Similarly, to show the zeros of q_n are distinct and in Y, we only need Y to be an interval.

In the above theorems we assume that w is STP or SSR. The proof shows that we can relax this condition somewhat. Indeed, for the existence of the biorthogonal polynomials, we only need that

$$\int_{x_0<x_1<\cdots<x_n}\int_{y_0<y_1<\cdots<x_n}\prod_{j<k}(x_k-x_j)\prod_{j<k}(y_k-y_j) \times \det[w(x_j,y_k)]_{j,k=0,\ldots,n}\,\mathrm{d}x_0\cdots\mathrm{d}x_n\,\mathrm{d}y_0\cdots\mathrm{d}y_n \tag{2.2}$$

does not vanish, where $x_i \in X$ and $y_i \in Y$ in the domain of the integrals. Furthermore, for theorem 2.2 to hold, what we need is that the integral

$$\int_{y_0<y_1<\cdots<y_m}\det[w(x_k,y_j)]_{j,k=0,\ldots,m}\prod_{i<l}(y_l-y_i)\,\mathrm{d}y_0\cdots\mathrm{d}y_m \tag{2.3}$$

does not vanish for any given $x_0 < x_1 < \cdots < x_n$, $x_i \in X$, and the integral

$$\int_{x_0<x_1<\cdots<x_m}\det[w(x_k,y_j)]_{j,k=0,\ldots,m}\prod_{i<l}(x_l-x_i)\,\mathrm{d}x_0\cdots\mathrm{d}x_m \tag{2.4}$$

does not vanish for any given $y_0 < y_1 < \cdots < y_n$, $y_i \in Y$. Therefore, if w is TP or SR and makes the above three integrals nonzero, then the theorems still hold. For example, we have the following result.

Theorem 2.3. *Let X and Y be open intervals. Let $w(x,y)$ be a function defined on $X \times Y$ such that w is* TP *or* SR *and all moments $m_{i,j}$ of w exist. If the above three integrals do not vanish, then the biorthogonal polynomials p_n and q_n are uniquely determined by (2.1) and all the zeros of p_n and q_n are distinct and inside X and Y, respectively.*

We note that w is TP or RS without the additional assumption is not enough. Indeed, consider the weight function $w(x,y) = u(x)v(y)$, where u and v are positive functions so that w has all finite moments; evidently, the matrix $[w(x_k,y_j)]_{j,k=0,\ldots,n}$ is of rank 1, and the determinant is zero for all $x_k \in X$ and $y_j \in Y$ for $n > 1$. The product weight function is TP but not STP. It is easy to see that biorthogonal polynomials do not exist for this weight function.

There are many examples of TP and SR functions that satisfy the conditions in the above theorems. Moreover, there are several properties that are useful for constructing a large class of such functions.

Proposition 2.4. *Let $w(x, y)$ be* SR *on $X \times Y$. (a) If $u(x)$, $v(y)$ are nonzero functions maintaining the same constant sign for $x \in X$ and $y \in Y$, respectively, then the function*

$$a(x, y) = u(x)v(y)w(x, y) \qquad x, y \in X \times Y$$

is SR. *(b) If $u = \phi^{-1}(x)$ and $v = \psi^{-1}(x)$, each defines a strictly increasing (decreasing) function mapping X and Y into U and V, respectively, where ϕ^{-1} and ψ^{-1} are the inverse functions of ϕ and ψ, then the function*

$$b(u, v) = w(\phi(u), \psi(v)) \qquad u, v \in U \times V$$

is SR *on U and V.*

This is [3, p 18, theorem 2.1], the proof follows trivially from the definition. The support set of a function $u(x)$ is the set of points such that $u(x) \neq 0$. As an immediate consequence of the above proposition, we state the following result.

Corollary 2.5. *If $w(x, y)$ satisfies the conditions of theorem 2.3, $u(x)$ and $v(y)$ are positive functions supported on X and Y, respectively, then the function $a(x, y) = u(x)v(y)w(x, y)$ satisfies the conditions of theorem 2.3 as well.*

Our second property deals with the convolution $w_1 * w_2$ defined by

$$(w_1 * w_2)(x, y) = \int_Z w_1(x, z)w_2(z, y)\, \mathrm{d}\mu(z) \qquad x \in X \quad y \in Y$$

where μ is a finite Borel measure defined on Y. We need the notation

$$w\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix} := \det[w(x_j, y_k)]_{k,j=1,\dots,n}.$$

The following is called the basis composition formula [3, p 17, (2.5)]:

$$\begin{aligned}(w_1 * w_2)&\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix} \\ &= \int_{z_1<z_2<\dots<z_n} \cdots \int w_1\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ z_1 & z_2 & \dots & z_n \end{pmatrix} w_2\begin{pmatrix} z_1 & z_2 & \dots & z_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix} \mathrm{d}z_1 \cdots \mathrm{d}z_n.\end{aligned}$$

This is a consequence of the Cauchy–Binet formula that is also used in [4]. In particular, using this formula repeatedly, it implies the following result on the convolution $(w_1 * w_2 * \dots * w_m)$ defined as in (1.5).

Proposition 2.6. *If $w_i(x, y)$ are* SR (SSR) *on $X_{i-1} \times X_i$, $1 \leqslant i \leqslant m$, and μ_i are positive Borel measures, then the function*

$$\begin{aligned}(w_1 * w_2 * \dots * w_m)&(x, y) \\ &= \int_{X_1\times\dots\times X_{m-1}} \cdots \int w_1(x, z_1)w_2(z_1, z_2)\cdots w_m(z_{m-1}, y)\, \mathrm{d}\mu_1(z_1)\cdots \mathrm{d}\mu_{m-1}(z_{m-1})\end{aligned}$$

is SR (SSR) *on $X \times Y$, where $X = X_0$ and $Y = X_m$.*

The following corollary is a consequence of the Fubini theorem and the above proposition:

Corollary 2.7. *If each weight function w_i satisfies the conditions of theorem 2.3 on $X \times Y$, then the weight function $w(x, y) = (w_1 * w_2 * \dots * w_m)(x, y)$ satisfies the conditions of theorem 2.3.*

It is worth noting that, for a fixed integer n, the proof of our results only requires $w(x, y)$ to be in SR_n or SSR_n. In particular, this means that we can discuss biorthogonal polynomials even when X and Y are finite sets, just as in the case of discrete polynomials. If X is a finite set, we use the notation $|X|$ to denote the number of elements in X.

Theorem 2.8. *Let X be a finite set and $N = |X|$, and let Y be either an infinite set or $|Y| \geqslant N$. Let $w(x, y)$ be a function defined on $X \times Y$ such that w is STP_N or SRP_N. Then the biorthogonal polynomials $\{p_n\}_{n=0}^N$ and $\{q_n\}_{n=0}^N$ are uniquely determined by relation (2.1).*

If Y is an interval, then we can also conclude that all zeros of q_n are real, distinct and lie inside Y.

3. Examples

3.1. Examples of weight functions

We are now ready to state our examples. Two of the simplest examples of the STP function are e^{-xy} for $x, y \in \mathbb{R}$ [3, p 15] and $1/(x + y)$ for $x, y \in (0, \infty)$ [3, p 149]. We note that if $w(x, y)$ is SR or SSR on $X \times Y$, then it is also SR or SSR on any subset of $X \times Y$. Hence, as a consequence of corollary 2.5, we have

Example 3.1. Let $u(x)$ be a positive function on (a, b) and let $v(y)$ be a positive function on (c, d). Assume all moments of the weight functions

$$w_1(x, y) = u(x)v(y)\,\mathrm{e}^{-xy} \qquad -\infty \leqslant a < b \leqslant \infty \quad -\infty \leqslant c < d \leqslant \infty$$

and

$$w_2(x, y) = u(x)v(y)/(x + y) \qquad 0 < a < b \leqslant \infty \quad 0 < c < d \leqslant \infty$$

exist. Then the monic biorthogonal polynomials p_n and q_n with respect to either w_1 or w_2 are uniquely determined and all the zeros of p_n are distinct and in (a, b) and all the zeros of q_n are distinct and in (c, d).

The weight function considered by Ercolani and McLaughlin [2] is the case $w_1(x, y)$ with $u(x) = \mathrm{e}^{-V(x)}$, $(a, b) = \mathbb{R}$ and $v(y) = \mathrm{e}^{-W(y)}$, $(c, d) = \mathbb{R}$. The case $w_2(x, y)$ with $u(x) = \mathrm{e}^{-x}$, $(a, b) = (0, \infty)$ and $v(y) = \mathrm{e}^{-y}$, $(c, d) = (0, \infty)$, is proved by Mehta [4]. Furthermore, corollary 3.2 shows that the conclusion also works for the convolution of the weight functions of the same type, as in [4]. The cases that (a, b) and (c, d) are finite intervals are permitted in the above example, for which all moments of w exist if w is a measurable function.

There are many other total positive or sign-regular functions that satisfy the condition of theorem 2.3. We list several examples below:

Example 3.2. Let ϕ and ψ be strictly increasing (decreasing) functions on (a, b) and (c, d), respectively, and $\phi : (a, b) \mapsto (A, B)$ and $\psi : (c, d) \mapsto (C, D)$. If $u(x)$ and $v(y)$ are positive functions on (a, b) and (c, d), respectively, and all the moments of

$$w_1(x, y) = u(x)v(y)\,\mathrm{e}^{-\phi(x)\psi(y)} \qquad x \in (a, b) \quad y \in (c, d)$$

or

$$w_2(x, y) = \frac{u(x)v(y)}{\phi(x) + \psi(y)} \qquad x \in (a, b) \quad y \in (c, d) \quad A, C > 0$$

exist, then the biorthogonal polynomials p_n and q_n exist and all their zeros are distinct and lie in (a, b) and (c, d), respectively.

Evidently, this is a consequence of proposition 2.4. Examples include weight functions

$$w(x, y) = u(x)v(y)\,\mathrm{e}^{-x^m y^m} \qquad -\infty \leqslant a < x, y < b \leqslant \infty$$

where m is an odd integer, and for all $\alpha, \beta > 0$,

$$w(x, y) = \frac{u(x)v(y)}{x^\alpha + y^\beta} \qquad x \in (a, b) \quad y \in (c, d) \quad 0 < a < b \quad 0 < c < d \quad A, C > 0$$

in which u and v are positive functions so that all moments of w exist.

An important class of TP functions is the Pólya frequency functions studied by Schoenberg [3, ch 7]. A Pólya frequency function of order r (PF$_r$) is a function f defined on $\mathbb{R}$ for which $w(x, y) = f(x - y)$ is TP$_r$. Again we use the notation PF if $r = \infty$. The class of PF is completely determined and all functions of the form $u(x)v(y)f(x - y)$, $f \in PF$, that have finite moments satisfy the condition of theorem 2.3. We summarize this as the next example.

Example 3.3. Let $f(u)$ be defined through its Laplace transform by

$$\int_{-\infty}^{\infty} \mathrm{e}^{-sx} f(x)\,\mathrm{d}x = \frac{1}{\mathrm{e}^{-\gamma s^2+\delta s} \prod_{i=1}^{\infty}(1 + a_i s)\,\mathrm{e}^{-a_i s}}$$

where $\gamma \geqslant 0$, δ real and $0 < \gamma + \sum a_i^2 < \infty$. If u and v are positive functions on $\mathbb{R}$ such that all the moments of the weight function $w(x, y) = u(x)v(y)f(x - y)$ exist, then $w(x, y)$ satisfies the condition of theorem 2.3.

This follows from proposition 2.4 and [3, p 357, theorem 6.1]; moreover, if $\gamma > 0$ then w is strictly PF. The simplest examples of f are $f(t) = \mathrm{e}^{-\gamma t^2}$, $\gamma > 0$, and

$$f(t) = \begin{cases} \mathrm{e}^{-\lambda t} & t \geqslant 0 \\ 0 & t < 0 \end{cases} \qquad \text{or} \qquad f(t) = \begin{cases} \mathrm{e}^{\lambda t} & t \leqslant 0 \\ 0 & t > 0 \end{cases}$$

where $\lambda > 0$. We note that the multiplication of $u(x)$ and $v(y)$ is necessary for all f in PF, since functions of the form $w(x, y) = f(x - y)$ do not have all finite moments as one can easily see in the example of $f(t) = \mathrm{e}^{-\gamma t}$.

3.2. Biorthogonality of the classical orthogonal polynomials

As pointed out in [2], in the case of

$$w(x, y) = \mathrm{e}^{-\alpha x^2}\mathrm{e}^{-\beta y^2}\mathrm{e}^{-xy} \qquad \alpha, \beta > 0 \quad \alpha\beta > 1 \qquad x, y \in \mathbb{R}$$

($w_1(x, y)$ in example 3.1 with $V(x) = \alpha x^2$ and $U(y) = \beta y^2$), the biorthogonal polynomials p_n and q_n are in fact Hermite polynomials $H_n(x)$ with a proper dilation,

$$p_n(x) = H_n\left(\frac{x}{\sqrt{\beta}}\sqrt{\alpha\beta - 1}\right) \qquad q_n(y) = H_n\left(\frac{y}{\sqrt{\alpha}}\sqrt{\alpha\beta - 1}\right).$$

Here and in the following, when the biorthogonal polynomials are in terms of orthogonal polynomials, they may not be normalized to be monic polynomials.

In the following we show several other examples in which classical orthogonal polynomials appear with certain biorthogonality.

As our first example, we consider the function defined by [3, p 16]

$$w(x, y) = \begin{cases} 1 & a \leqslant x \leqslant y \leqslant b \\ 0 & a \leqslant y < x \leqslant b. \end{cases} \tag{3.1}$$

This function is TP. Furthermore, for arbitrary $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$, we have

$$w\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix} = \begin{cases} 1 & x_1 \leqslant y_1 < x_2 \leqslant y_2 < \cdots < x_n \leqslant y_n \\ 0 & \text{otherwise.} \end{cases}$$

Evidently $w(x, y)$ satisfies the conditions of theorem 2.3, so that the biorthogonal polynomials with respect to w exist. It turns out that these polynomials are in fact certain Jacobi polynomials. The Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, are classical orthogonal polynomials orthogonal with respect to the weight function

$$u^{(\alpha,\beta)}(x) = (1-x)^\alpha(1-x)^\beta \qquad -1 < x < 1 \quad \alpha > -1 \quad \beta > -1.$$

Let us assume that $P_n^{(\alpha,\beta)}$ denote the monic polynomials. A simple change of the variable gives the corresponding weight function on (a, b). To simplify the notation, we work with $(a, b) = (-1, 1)$ and we consider a slightly more general weight function.

Proposition 3.4. *Let* $w(x, y)$ *be defined as in (3.1) on* $(-1, 1)$. *Then the biorthogonal polynomials* p_n *and* q_n *with respect to the weight function*

$$a(x, y) = (1-x)^\alpha(1+y)^\beta w(x, y) \qquad -1 < x, y < 1 \quad \alpha, \beta > -1$$

are in fact orthogonal polynomials, $p_n(x) = P_n^{(\alpha,\beta+1)}(x)$ *and* $q_n(y) = P_n^{(\alpha+1,\beta)}(y)$.

Proof. From the definition of $w(x, y)$ it follows that

$$\begin{aligned}\phi_m(x) &:= \int_{-1}^{1}(1+y)^m a(x, y)\,\mathrm{d}y = (1-x)^\alpha\int_{-1}^{x}(1+y)^{m+\beta}\,\mathrm{d}y \\ &= (1-x)^\alpha(1+x)^{m+\beta+1}/(m+\alpha+1).\end{aligned}$$

The biorthogonality of $p_n(x)$ to $q_m(y)$ implies that

$$\int_{-1}^{1} p_n(x)\phi_m(x)\,\mathrm{d}x = \int_{-1}^{1}\int_{-1}^{1} p_n(x)(1+y)^m a(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0$$

for $0 \leqslant m \leqslant n-1$. Consequently, the above formula of ϕ_m shows that p_n is orthogonal to $(1+y)^m$ with respect to the weight function $(1-x)^\alpha(1+x)^{\beta+1}$, so that $p_n(x)$ is equal to $P_n^{(\alpha,\beta+1)}$. Working with the integral of $(1-x)^m$ with respect to $a(x, y)$, the proof for $q_n(y)$ follows similarly. □

We could also multiply the weight function in this example by $(y-x)_+^\gamma$, which is equal to $(y-x)^\gamma$ if $y > x$ and zero otherwise, to get biorthogonality of the other pair of Jacobi polynomials.

In the definition of the biorthogonal polynomials, we can also take X and Y as discrete sets, for example, the set of non-negative integers $\mathbb{N}_0$. Our next example examines the weight function related to the Poisson distribution. In this case we run into Laguerre polynomials, $L_n^\alpha(x)$, which are orthogonal with respect to $x^\alpha\,\mathrm{e}^{-x}$ on $[0, \infty)$ and a special case of the Mexiner polynomials, $M_n(x; b, c)$, whose orthogonal relation is given by [1, p 346]

$$\sum_{x=0}^{\infty}\frac{(b)_x}{x!}c^x M_m(x; b, c)M_n(x; b, c) = \frac{c^{-n}n!}{(b)_n(1-c)^b}\delta_{m,n}$$

where $(a)_m$ denote the Pochhammer symbol $(a)_m = a(a+1)\cdots(a+m-1)$.

Proposition 3.5. *Let* $c > 0$ *and* $\alpha > -1$ *be two constants, and let* $w(x, y)$ *be defined by*

$$w(x, y) = \frac{1}{\Gamma(x+1)}(cy)^x y^\alpha\,\mathrm{e}^{-y} \qquad y \in (0, \infty) \quad x \in \mathbb{N}_0.$$

Then the biorthogonal polynomials p_n and q_n determined by

$$\int_0^\infty \sum_{x=0}^\infty p_n(x) q_m(y) w(x,y)\,\mathrm{d}y = h_n \delta_{n,m} \qquad h_n \neq 0$$

are orthogonal polynomials, $p_n(x) = M_n(x; \alpha+1, c)$ and $q_n(y) = L_n^\alpha((1+c)x)$.

Proof. Since the weight function $t^x = \mathrm{e}^{x \ln t}$ is STP, the weight function $w(x,t)$ is STP by proposition 2.4. By the definition of the gamma function, the fact that p_n is biorthogonal to y^m, $0 \leqslant m \leqslant n-1$, gives

$$\begin{aligned} 0 &= \sum_{x=0}^\infty p_n(x) \int_0^\infty y^m w(x,y)\,\mathrm{d}y \\ &= \sum_{x=0}^\infty p_n(x) \frac{1}{\Gamma(x+1)} c^x \int_0^\infty y^{m+x+\alpha}\,\mathrm{e}^{-y}\,\mathrm{d}y = \sum_{x=0}^\infty p_n(x) \frac{\Gamma(m+x+\alpha+1)}{\Gamma(x+1)} c^x. \end{aligned}$$

Using $(\alpha+1)_x = \Gamma(\alpha+x+1)/\Gamma(\alpha+1)$ we have

$$\frac{\Gamma(m+x+\alpha+1)}{\Gamma(x+1)} = \frac{\Gamma(m+x+\alpha+1)}{\Gamma(x+\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(x+1)} (\alpha+1)_x;$$

since $\Gamma(m+x+\alpha+1)/\Gamma(x+\alpha+1)$ is a polynomial of degree m in x, it follows that $p_n(x)$ is orthogonal to polynomials of degree at most $n-1$ with respect to the discrete measure $c^x(\alpha+1)_x/\Gamma(x+1)$. Hence, p_n is the Mexiner polynomial $M_n(x; \alpha+1, c)$.

The biorthogonal relation of $q_n(y)$ orthogonal to x^m, $0 \leqslant m \leqslant n-1$, shows that $q_n(y)$ is orthogonal to ψ_m defined by

$$\psi_m(y) = \sum_{x=0}^\infty x^m (ct)^x / x!$$

for $0 \leqslant m \leqslant n-1$. It is easy to see that $\psi_0(y) = \mathrm{e}^{-cy}$. Moreover, since

$$\psi_m(y) = \sum_{x=1}^\infty x^{m-1} (cy)^x/(x-1)! = cy \sum_{x=0}^\infty (x+1)^{m-1} (cy)^x / x!$$

it follows from induction that $g_m(y) = \mathrm{e}^{cy}\psi_m(y)$ is a polynomial in y of degree m. Consequently,

$$0 = \int_0^\infty q_n(y)\psi_m(y) y^\alpha\,\mathrm{e}^{-y}\,\mathrm{d}y = \int_0^\infty q_n(y) g_m(y) y^\alpha\,\mathrm{e}^{-(1+c)y}\,\mathrm{d}y$$

so that a simple change of the variable $y \mapsto t/(1+c)$ shows that $q_n(t/(1+c))$ is orthogonal to polynomials of degree at most $n-1$ with respect to $t^\alpha\,\mathrm{e}^{-t}$. Hence, $q_n(y) = L_n^\alpha((1+c)t)$. □

In our last example, the weight function is related to the binomial distribution and one of the variables is defined on a finite set. We will need the Hahn polynomials, $Q_n(x; \alpha, \beta, N)$, whose orthogonal relation is given by [1, p 345]

$$\sum_{x=0}^N \frac{(\alpha+1)_x (\beta+1)_{N-x}}{x!(N-x)!} Q_n(x; \alpha, \beta, , N) Q_m(x; \alpha, \beta, , N) = h_n \delta_{m,n}$$

where $\alpha, \beta > -1$, $h_n \neq 0$ can be explicitly given but will not be needed below.

Proposition 3.6. *Let N be a positive integer and let $w(x, y)$ be defined by*

$$w(x,y) = \binom{N}{x} y^x (1-y)^{N-x} \qquad 0 < y < 1 \quad X = 0, 1, \ldots, N.$$

Then the biorthogonal polynomials p_n and q_n determined by

$$\int_0^1 \sum_{x=0}^{N} p_n(x) q_m(y) w(x,y)\, \mathrm{d}y = h_n \delta_{n,m} \qquad h_n \neq 0 \quad 0 \leqslant m, n \leqslant N$$

are orthogonal polynomials, $p_n(x) = Q_n(x; 0, 0, N)$ and $q_n(y) = P_n^{(0,0)}(2y-1)$.

Proof. Changing the variable $y \mapsto \mathrm{e}^t/(1+\mathrm{e}^t)$, then $w(x, y)$ becomes $\binom{N}{x}\mathrm{e}^{xt}(1+\mathrm{e}^t)^{-N}$ as indicated in [3, p 10]. Hence, using the fact that e^{xt} is STP and proposition 2.4, the weight function is STP_N. Hence, theorem 2.8 shows that p_n and q_n exist for $0 \leqslant n \leqslant N$.

Using the beta integral, the biorthogonality of $p_n(x)$ to y^m, $0 \leqslant m \leqslant n-1$, shows that

$$\begin{aligned} 0 &= \sum_{x=0}^{N} \int_0^1 p_n(x) y^m w(x,y)\, \mathrm{d}y = \sum_{x=0}^{N} \binom{N}{x} p_n(x) \int_0^1 y^{m+x}(1-y)^{N-x}\, \mathrm{d}y \\ &= \sum_{x=0}^{N} \binom{N}{x} p_n(x) \frac{(m+x)!(N-x)!}{(m+N)!} = \frac{N!}{(N+m)!} \sum_{x=0}^{N} p_n(x) \frac{(m+x)!}{x!}. \end{aligned}$$

Since $(m+x)!/x!$ is a polynomial of degree m in x, this shows that $p_n(x)$ is orthogonal with respect to lower degree polynomials with respect to the unit weight. Hence, $p_n(x) = M_n(x; 1, 1)$ since $(1)_x = x!$.

On the other hand, the orthogonality of $q_n(y)$ to x^m, $0 \leqslant m \leqslant n-1$, shows that q_n is orthogonal to

$$\psi_m(y) = \sum_{x=0}^{N} x^m \binom{N}{x} y^x (1-y)^{N-x} \qquad 0 \leqslant m \leqslant n-1$$

with respect to the unit weight function on [0, 1]. Since it is easy to see that

$$\psi_m(y) = yN \sum_{x=0}^{N-1} (x+1)^{m-1} \binom{N-1}{x} y^x (1-y)^{N-x-1}$$

induction shows that $\psi_m(y)$ is a polynomial of degree m in y, so that $q_n(y)$ is orthogonal to lower degree polynomials with respect to $\mathrm{d}y$ on [0, 1]. □

The usual binomial distribution is associated with the Krawtchouk polynomials, $K_n(x; p, N)$, whose orthogonal relation is given by [1, p 347]

$$\sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x} K_n(x; p, N) K_m(x; p, N) = \frac{(-1)^n n!}{(-N)_n} \left(\frac{1-p}{p}\right)^n \delta_{m,n}$$

where $0 < p < 1$. It is interesting to note that a special case of the Hahn polynomial, not the Krawtchouk polynomial, appears in the biorthogonal relation with respect to the binomial distribution.

4. Conclusion

Using the concept of total positive functions or sign-regular functions, we showed that the argument of Ercolani and McLaughlin in [2] can be extended to large classes of weight functions, so that the biorthogonal polynomials exist and have real and distinct zeros.

Many examples are presented, including several whose biorthogonal polynomials are classical orthogonal polynomials which give new orthogonal relations between different families of classical orthogonal polynomials.

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